SI112: Advanced Geometry	Spring 2018
Lecture 22 — May 15th, Tuesday	
Prof. Manolis	*Scribe*: Liangzu

1 Lecture 22

1.1 Overview of This Lecture

It takes us almost the whole lecture (i.e., lecture 22) to prove irreducible decomposition theorem.

1.2 Proof of Things

Lemma 1.2.1. *Let Σ be the set of all closed set in $\mathbb{F}^n$ and G a subset of Σ. Then G contains a minimal element.*

Proof. Let $G' = \{I_Y : Y \in G\}$. Then G' is a set of (vanishing) ideals, where the ideals are in the Noetherian ring $\mathbb{F}[x_1, x_2, \dots, x_n]$, which means that there is a maximal element I_{Y^*} in G'. We will show that Y^* is a minimal element in G. Let $Y \subset Y^* \in G$. Then

$$I_{Y^*} \subset I_Y \in G' \Rightarrow I_{Y^*} = I_Y \Rightarrow Y^* = \overline{Y^*} = Z(I_{Y^*}) = Z(I_Y) = \overline{Y} = Y.$$

This proves that Y^* is minimal in G. □

Theorem 1.2.2 (irreducible decomposition theorem, Hartshorne\Prp I.1.5, p5)**.** *Let Y be a nonempty closed set of $\mathbb{F}^n$ ($\mathbb{F}^n$ is an infinite field). Then Y can be uniquely written as $Y = Y_1 \cup \cdots \cup Y_n$, where Y_i's are irreducible closed sets and $Y_i \not\subset Y_j$ for $i \neq j$.*

Proof. We first prove the existence and then the uniqueness. For uniqueness part a wrong proof is additionally given.

- **Existence.** Let G be the set of nonempty closed subsets of $\mathbb{F}^n$ that can not be written as a finite union of irreducible closed subsets. It is enough to show $G = \emptyset$, from which we will know that every closed subsets of $\mathbb{F}^n$ can be decomposed. Then there are three immediate observations to understand the structure of G.

  1. There is a minimal element Y^* in G by Lemma 1.2.1.
  2. every nonempty closed set Y in G is not irreducible, for otherwise $Y = Y$ is a unique irreducible decomposition.

3. every nonempty closed set X, which is not in G, can be (uniquely) written as a finite union of irreducible closed subsets $X_1 \cup X_2 \cup \cdots \cup X_n$.

Then, specifically, the set Y^* as a minimal element in G is not irreducible, i.e., $Y^* = Y \cup Y'$ for some Y, Y' being nonempty proper closed subsets of Y^*. Hence $Y, Y' \notin G$ by the minimality of Y^*. Observation 3. tells us that there exist some $Y_1, Y_2, \ldots, Y_n, Y_1', Y_2', \ldots, Y_m'$ such that $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ and $Y' = Y_1' \cup Y_2' \cup \cdots \cup Y_m'$. This implies

$$Y^* = Y_1 \cup Y_2 \cup \cdots \cup Y_n \cup Y_1' \cup Y_2' \cup \cdots \cup Y_m',$$

contradicting to the construction of G. Hence $G = \emptyset$.

We conclude that every closed set Y in $\mathbb{F}^n$ can be written as a union $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ of irreducible subsets. By throwing away a few if necessary, we may assume $Y_i \not\subset Y_j$ for $i \neq j$.

- **Uniqueness.** Now suppose $Y = Y_1' \cup Y_2' \cup \cdots \cup Y_m'$ is another such representation and $n \leq m$. Then we have

$$Y_1 \subset Y = Y_1' \cup Y_2' \cup \cdots \cup Y_m' \Rightarrow Y_1 = \cup_{i=1}^m (Y_1 \cap Y_i').$$

But Y_1 is irreducible, hence

$$Y_1 = Y_1 \cap Y_j' \iff Y_1 \subset Y_j'$$

for some $j \in \{1, 2, \ldots, m\}$, say without loss of generality $j = 1$. Similarly we have $Y_1' \subset Y_i$ for some $i \in \{1, 2, \ldots, n\}$. Hence $Y_1 (\subset Y_1') \subset Y_i$. But $Y_i \not\subset Y_j$ for $i \neq j$. This implies $i = 1$. Then $Y_1 = Y_1'$. Going deeper, we can do the similar for Y_2 to obtain $Y_2 = Y_2'$. Indeed,

$$Y_2 \subset Y = Y_1' \cup Y_2' \cup \cdots \cup Y_m' \Rightarrow Y_2 = \cup_{i=1}^m (Y_2 \cap Y_i'),$$

which, by irreducibility of Y_2, means

$$Y_2 = Y_2 \cap Y_j' \iff Y_2 \subset Y_j'$$

for some $j \in \{2, 3 \ldots, m\}$ (j can not be 1, for otherwise $Y_2 \subset Y_1' = Y_1$), say $j = 2$. Then similarly we have $Y_2' \subset Y_i$ for some $i \in \{2, 3, \ldots, n\}$, which means $i = 2$ and thus $Y_2 = Y_2'$. Continuing in this way we have $Y_i = Y_i'$ for $i = 1, 2, \ldots n$. Then we have $n = m$ for a similar reason (Consider Y_{n+1}' if $n < m$).

□

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2 Lecture 23

Section 8.1 and 8.2 in the book Ideals, Varieties, and Algorithms give rich examples for projective spaces and varieties, though a little bit wordy.

2.1 Further Reading

- Section 8.1, 8.2 in this book: Ideals, Varieties, and Algorithms.