SI112: Advanced Geometry	Spring 2018
Lecture Note 9 — Mar. 27th, Tuesday	
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1 Lecture 9

1.1 Overview of This Lecture

In this lecture we introduce an important concept: *compactness*. After introducing its definition (1.2.4), we devolops theorems, as usual, that relate compactness and other important topological concepts, e.g., compactness in relative topology (1.2.8), neighborhood (1.2.10), continuity (1.2.12) and closedness (1.2.15,1.2.17).

We will spend 2 lectures on compactness (this and the next lecture).

1.2 Proof of Things

Definition 1.2.1 (covering, definition 5.2.1)**.** Let X be a set, B a subset of X, and $\{A_i\}_{i\in I}$ is called a *covering* of B or is said to *cover* B if $B \subset \cup_{i\in I} A_i$. If, in addition, the indexing set I is finite, $\{A_i\}_{i\in I}$ a *finite covering* of B.

Definition 1.2.2 (subcovering, definition 5.2.2)**.** Let X be a set and let $\{A_i\}_{i\in I}, \{B_k\}_{k\in J}$ be two coverings of a subset C of X. If for each $i \in I, A_i = B_k$ for some $k \in J$, then the covering $\{A_i\}_{i\in I}$ is called a *subcovering* of the covering $\{B_k\}_{k\in J}$. Note that this definition is not introduced in the lecture.

Exercise 1.2.3 (open covering, definition 5.2.3)**.** An *open covering* of a set B is a union of open set which covers B. Try to give it a rigorous definition. Or have a look at definition 5.2.3 in Mendelson.

Definition 1.2.4 (definition 5.2.4)**.** A topological space X is said to be *compact* if for each open covering $\{U_i\}_{i\in I}$ of X there is a finite subcovering $U_{i_1}, \dots, U_{i_n}$.

Remark 1.2.5 (remark for definition 1.2.4)**.** Compactness allows to study global properties by looking at a finite number of neighborhood. Well, the concept of compactness is somewhat elusive and unmotivated. Have a look at this paper if you are interested in.

Remark 1.2.6 (remark for definition 1.2.4)**.** Given the definition of compactness, how to prove a given set, say X, is compact or not? To prove X is compact, you need to show that **for each** *open* covering of X, there is a *finite* subcovering. To prove that X is not compact, in contrast, you need to give an counterexample, i.e., there exists a open covering of X such that there are no subcoverings.

Definition 1.2.7 (definition 5.2.5)**.** A subset C of a topological space X is said to be *compact*, if C is a compact topological space in the relative topology.

Exercise 1.2.8 (theorem 5.2.6)**.** Prove it: A subset C a topological space X is compact if and only if for each open covering $\{U_i\}_{i\in I}$, U_i open in X, there is a finite subcovering $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ of C.

Remark 1.2.9 (remark for exercise 1.2.8)**.** This exercise is theorem 5.2.6 in Mendelson. We skipped it in this lecture. You can prove it by yourself. Use the definition of relative topology and compactness.

Theorem 1.2.10 (theorem 5.2.7)**.** *A topological space X is compact if and only if, whenever for each $x \in X$ a neighborhood N_x of x is given, there is a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i}$.*

Proof. On the one hand, suppose X is compact. For each $x \in X$ there is a neighborhood N_x of x (why?). Hence for each x, there is an open set U_x such that $x \in U_x \subset N_x$ and $\{U_x\}_{x\in X}$ is an open covering of X. Since X is compact there is a finte subcovering $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$, i.e., $X = \cup_{i=1}^n U_{x_i}$. But $U_{x_i} \subset N_{x_i}$ for each i, hence $X = \cup_{i=1}^n N_{x_i}$.

On the other hand, suppose whenever for each $x \in X$ a neighborhood N_x of x is given, there is a finite number of points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i}$. We want to show that X is compact. The below is a **wrong** proof.

For each $x \in X$ there is an open set O_x in X containing x (why?), which is a neighborhood N_x of x, then we have $X = \cup_{x\in X} O_x = \cup_{x\in X} N_x$. By our hypothesis, there are points $x_1, x_2, \ldots, x_n$ of X such that $X = \cup_{i=1}^n N_{x_i} = \cup_{i=1}^n O_{x_i}$. Hence X is compact.

Why is this proof wrong? The problem here is that we have to start with an **arbitrary** open covering $\{U_i\}_{i\in I}$ of X, then we need to show that there is a finite subcovering. Since $\{U_i\}_{i\in I}$ covers X, for each $x \in X$ we have $x \in U_i$ for some $i \in I$. Notice here that different x can be in the same U_i, i.e., it is possible that $x_1, x_2 \in X$ and $x_1 \in U_i, x_2 \in U_i$ for some $i \in I$. To rephrase, for each $x \in X$, there is an $i = i(x)$ such that $x \in U_i$, which is a neighborhood of x. Let $N_x = U_i$, then by our hypothesis, there are points $x_1, x_2, \ldots, x_n$ of X such that $N_{x_i} = U_{i(x_i)}, i = 1, 2, \ldots, n$ covers X, and hence X is compact. □

Theorem 1.2.11 (theorem 5.2.8)**.** *A topological space is compact if and only if whenever a family $\cap_{i\in I} A_i = \emptyset$ of closed sets is such that $\{A_i\}_{i\in I}$ then there is a finite subset of indices $\{i_1, i_2, \ldots, i_n\}$ such that $\cap_{k=1}^n A_{i_k} = \emptyset$.*

proof skeleton. Use the definition of compactness and "the complement of a closed set is open". □

Theorem 1.2.12 (theorem 5.2.9)**.** *Let $f : X \to Y$ be continuous and let A be a compact subset of X. Then $f(A)$ is a compact subset of Y.*

Proof. This theorem shows that continuous functions preserve compactness.

To show that $F(A)$ is a compact subset of Y, let's start with an arbitrary open covering $\{V_i\}_{i\in I}$ of $f(A)$, i.e., $f(A) \subset \cup_{i\in I} V_i$. Then we have $A \subset f^{-1}(f(A)) \subset \cup_{i\in I} f^{-1}(V)$, which means that $\{f^{-1}(V_i)\}_{i\in I}$ is a covering of A. In addition, since f is continuous and V_i is open for each $i \in I$, $\{f^{-1}(V_i)\}_{i\in I}$ is an open covering of A. Since A is compact, there is a finite subcovering $f^{-1}(V_{i_1}), f^{-1}(V_{i_2}), \dots, f^{-1}(V_{i_n})$ of A, i.e., $A \subset \cup_{k=1}^{n} f^{-1}(V_{i_k})$.

Remember that we want to show that there is a finite covering of $f(A)$. By theorem 1.2.8, it is enough to show that $f(A) \subset \cup_{k=1}^{n} V_{i_k}$. Does $A \subset \cup_{k=1}^{n} f^{-1}(V_{i_k})$ imply $f(A) \subset \cup_{k=1}^{n} V_{i_k}$? prove it!

□

Corollary 1.2.13 (corollary 5.2.10)**.** *Let the topological spaces X and Y be homeomorphic, then X is compact if and only if Y is compact.*

Example 1.2.14. The open interval $(0, 1)$ is not compact. To show this, we need to construct a covering of $(0, 1)$ that does not have a finite subcovering. (*hint:* construct something like $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$.)

Theorem 1.2.15 (theorem 5.2.11)**.** *Let X be compact and A closed in X. Then A is compact.*

Proof. Let $\{V_i\}_{i\in I}$ be an open covering of the closed set A, i.e., $A \subset \cup_{i\in I} V_i$. Then

$$X = A \cup A^C = \cup_{i\in I} V_i \cup A^C .$$

Since X is compact, there is a subcovering $U_{i_1}, \dots, U_{i_n}$, i.e., $X = \cup_{k=1}^{n} U_{i_k}$, where for each k, $U_{i_k} = V_i$ for some $i \in I$, or $U_{i_k} = A^C$. Is $U_{i_1}, \dots, U_{i_n}$ a finite subcovering of A? Why? How can we finish the proof? □

Lemma 1.2.16 (lemma for theorem 1.2.17)**.** *In a topological space X, the intersection of a finite set of neighborhoods of a point x is a neighborhood of x.*

proof skeleton. Immediate. Apply the definition of neighborhood. □

Theorem 1.2.17 (theorem 5.2.12)**.** *Let X be a Hausdorff Space. If a subset F of X is compact, then F is closed.*

Proof. This is a theorem that requires us to prove again that some set F is closed. Review how we prove a set is closed in lecture 4.

So, to prove F is closed, it is enough to show that there are no limit points of F in F^C (why?). Hence we need to prove the following:

$$\forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap F = \emptyset. \tag{1}$$

Now let's consider the compact set F. For each $x \in F$, let V_x be an open set containing x, then we have $F = \cup_{x \in F} V_x$, then there is a subcovering $V_{x_1}, \ldots, V_{x_n}$, i.e., $F = \cup_{i=1}^n V_{x_i}$. Hence we need to prove

$$\begin{aligned} & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap (\cup_{i=1}^n V_{x_i}) = \emptyset \\ \iff & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } \cup_{i=1}^n (N \cap V_{x_i}) = \emptyset \\ \iff & \forall z \in F^C, \text{ there is a neighborhood } N \text{ of } z \text{ such that } N \cap V_{x_i} = \emptyset, \forall i = 1, \ldots, n. \end{aligned} \tag{2}$$

Let $z \in F^C$ be given. For each $i = 1, \ldots, n$, there exists a neighborhood N_{x_i} of z such that $V_{x_i} \cap N_{x_i} = \emptyset$ (why?). Let $N = \cap_{i=1}^n N_{x_i}$, then N is a neighborhood of z (by lemma 1.2.16), and $N \cap V_{x_i} = \emptyset, \forall i = 1, \ldots, n$. We finished the proof.

□

Definition 1.2.18 (bounded, definition 5.3.1)**.** A subset A of $\mathbb{R}^n$ is said to be *bounded* if there is a real number K such that for each $x = (x_1, x_2, \ldots, x_n) \in A, |x_i| \leq K, i = 1, \ldots, n$.

Lemma 1.2.19 (lemma 5.3.2)**.** *If A is a compact subset of $\mathbb{R}$ then A is closed and bounded.*

proof skeleton. It is easy to prove by theorem 1.2.17 that A is closed. To prove A is bounded, you need to construct an open covering of A, which will be reduced to a finite subcovering. Note that, based on your construction, the finite subcovering is basically a collection of open intervals. Now show that A is bounded. □

1.3 Further Reading

5.1-5.4 in Mendelson.

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2 Lecture 10

Q: "Why don't you use different colors"?

A: "It is not professional".

2.1 Overview of This Lecture

Lecture 10 is a continuation of our exploration on compactness. Some theorems of great relevance are introduced. However, some of the proofs are quite challenging and elusive.

2.2 Proof of Things

Lemma 2.2.1 (lemma 5.3.3)**.** *The closed interval* $[0, 1]$ *is compact.*

Proof. I read many proofs for this problem (this, and that), but none of them are rigorous. What's wrong with them? (read them, they are good lessons.)

Let $\{U_i\}_{i\in I}$ be any open covering of $[0, 1]$. The trick is to consider the set

$$A = \{x \in [0, 1] : [0, \text{x}] \text{ can be covered by finitely many of the } U_i\text{'s}\} .$$

Note that A is non-empty and bounded, and $0 \in A$. Then use the completeness property of $\mathbb{R}$ to take s be the least upper bound of A. Note also that $0 \le s \le 1$.

We first show that $0 < s$. Since $0 \in U_i$ for some $i \in I$ and U_i is open, U_i is a neighborhood of 0. It follows that there is an open set $(-\epsilon, \epsilon)$ such that $0 \in (-\epsilon, \epsilon) \subset U_i$, which implies $[0, \frac{\epsilon}{2}]$ can be covered by U_i. Hence $0 < \frac{\epsilon}{2} \le s$, i.e., $0 < s$.

We then show that for each $0 \le t < s$, $[0, t]$ can be covered by finitely many sets in A, i.e.,$t \in A$. Suppose for the sake of contradiction $t \notin A$. Then every point $x \in A$ has to satisfy $x < t$, for if $x \ge t$ and $x \in A$, then finitely many open sets covering $[0, x]$ can also cover $[0, t]$. It follows that t is an upper bound for A, but $t < s$, contradicting the choice of s. Consequently, we have $[0, s) \subset A$.

Finally we will show that $s = 1$, which will prove that $[0, 1]$ is compact. Suppose for the sake of contradiction $s < 1$. There is $U_0 \in \{U_i\}_{i\in I}$ such that $s \in U_0$, which means that there is an open set $(s - \epsilon, s + \epsilon)$ such that $s \in (s - \epsilon, s + \epsilon) \subset U_0$. Note that $s - \epsilon \in A$, that is, $[0, s - \epsilon]$ can be covered by finitely many open sets, say $U_{i_1}, U_{i_2}, \dots, U_{i_n}$. Then the open sets,

$U_{i_1}, U_{i_2}, \ldots, U_{i_n}$, together with the open set U_0, will cover $[0, s+\frac{\epsilon}{2}]$, i.e., $s+\frac{\epsilon}{2} \in A$, which contradicts the definition of s. ☐

Corollary 2.2.2 (corollary 5.3.4)**.** *Each closed interval* $[a, b]$ *is compact.*

Proof. Immediate. ☐

Theorem 2.2.3 (theorem 5.3.5)**.** *A subset* A *of the real line is compact if and only if* A *is closed and bounded.*

Proof. Let A be compact. Then by lemma 1.2.19 A is closed and bounded.

Let A be closed and bounded. Then there exists $K > 0$ such that $A \subset [-K, K]$ and A is closed in the compact set $[-K, K]$. By theorem 1.2.15, A is compact. ☐

Definition 2.2.4 (basis for a topological space, definition 3.7.3)**.** Let X be a topological space and $\{B_i\}_{i \in I}$ a collection of open sets in X. $\{B_i\}_{i \in I}$ is called a *basis for the open sets of* X if each open set in X is a union of members of $\{B_i\}_{i \in I}$.

Remark 2.2.5. In lecture 8, we defined product topology without explicitly defining *basis*. It is recommended to read 3.7 in Mendelson before proceeding. You should verify that the sets of the form $O_1 \times O_2$, O_1, O_2 open in the topological spaces X_1, X_2 respectively, are a basis for the open sets of the topological space $X_1 \times X_2$.

Example 2.2.6. In $\mathbb{R}$ with standard topology a base for the topology is the collection of all open intervals.

Definition 2.2.7 (dimension of topological space)**.** Let X be a topological space, the dimension, denoted by $\dim X$ is the supremum among all lengths of chains $X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_i$ of closed and irreducible subsets of X.

Remark 2.2.8 (remark for definition 2.2.7)**.** This definition is irrelevant, at least for now.

Lemma 2.2.9 (lemma 5.4.1)**.** *Let* $\mathcal{B}$ *be a basis for the open sets of a topological space* Z*. If, for each covering* $\{\mathcal{B}_\beta\}_{\beta \in J}$ *of* Z *by members of* $\mathcal{B}$*, there is a finite subcovering, then* Z *is compact.*

Proof. Let $\{\mathcal{O}_i\}_{i \in I}$ be an open covering of Z, we need to show that there is a finite subcovering. Since $\mathcal{B}$ is a basis for the topologocal space Z, for each $i \in I$, there exists $J_i \subset J$ such that $O_i = \cup_{\beta \in J_i} \mathcal{B}_\beta$ and that $\mathcal{B}_\beta \subset O_i$ for each $\beta \in J_i$. Hence $Z \subset \cup_{i \in I} O_i = \cup_{i \in I} \cup_{\beta \in J_i} \mathcal{B}_\beta$. By the supposition, $Z \subset \cup_{k=1}^{n} \mathcal{B}_{\beta_k}$, where $\mathcal{B}_{\beta_k}$ is a subset of O_{β_k} for some $O_{\beta_k} \in \{\mathcal{O}_i\}_{i \in I}$. Consequently, $Z \subset \cup_{k=1}^{n} \mathcal{O}_{\beta_k}$. We finished the proof. ☐

Theorem 2.2.10 (theorem 5.4.2)**.** *Let X and X' be compact topological spaces, then $X \times X'$ is compact.*

Proof. The common strategy for proving compactness possibly fails to prove this theorem, which should not stop you having a try. A complete but not compact proof is given here.

As mentioned above, the set $\{O \times O' : O \text{ is open in } X \text{ and } O' \text{ is open in } X'\}$ is a basis for the open sets of the topological space $X \times X'$. By lemma 2.2.9, it is enough to show that each covering of $X \times X'$ by sets of the form $O \times O'$, O is open in X, O' is open in X', has a finite subcovering.

Let $\{O_i \times O_i'\}_{i\in I}$ be such a covering. Then for each $x_0 \in X$, the open covering $\{O_i \times O_i'\}_{i\in I}$ is necessarily an open covering of the set $X'_{x_0} = \{x_0\} \times X' = \{(x_0, x') : x' \in X'\}$. But X'_{x_0} is homeomorphic to X' and hence compact. There is therefore a finite subset I_{x_0} of I such that $\{O_i \times O_i'\}_{i\in I_{x_0}}$ covers X'_{x_0}.

Without loss of generality, we may assume that $x_0 \in O_j$ for each $j \in I_{x_0}$, for otherwise we may delete $O_j \times O_j'$ and still cover X'_{x_0} (why still cover X'_{x_0} after deleting?).

The set $O^*_{x_0} = \cap_{i\in I_{x_0}} O_i$ is a finite intersection of open sets containing x_0 and is therefore an open set containing x_0. Now we **claim** that $\{O_i \times O_i'\}_{i\in I_{x_0}}$ is an open covering of $O^*_{x_0} \times X'$. For each $(x, x') \in O^*_{x_0} \times X'$, we have that $x \in O^*_{x_0} = \cap_{i\in I_{x_0}} O_i$ and $x' \in X'$, which means that $x \in O_i$ for each $i \in I_{x_0}$ and $x' \in O_j'$ for some $j \in I_{x_0}$. Hence $(x, x') \in O_j \times O_j'$ for some $j \in I_{x_0}$.

Notice that $\{O^*_x\}_{x\in X}$ is an open covering of the compact space X, hence there is a finite subcovering $O^*_{x_1}, O^*_{x_2}, \dots, O^*_{x_n}$ of X. Let us set $I^* = I_{x_1} \cup I_{x_2} \cup \cdots \cup I_{x_n}$ and show that the finite family $\{O_i \times O_i'\}_{i\in I^*}$ is a covering of $X \times X'$, from which it will follow that $X \times X'$ is compact. Suppose $(x, x') \in X \times X'$. Since $\{O_i\}_{i\in I^*}$ covers X, $x \in O^*_{x_i}$ and $(x, x') \in O^*_{x_i} \times X'$ for some x_i. By our previous **claim**, $(x, x') \in O_j \times O_j'$ for some $j \in I_{x_i}$, which certainly implies that $(x, x') \in O_i \times O_i'$ for some $i \in I^*$. We have thus established that $\{O_i \times O_i'\}_{i\in I^*}$ is a finite covering of $X \times X'$ and that therefore $X \times X'$ is compact. □

Corollary 2.2.11 (corollary 5.4.3)**.** *Let $X_1, X_2, \dots, X_n$ be compact topological spaces. Then $\prod_{i=1}^n X_i$ is also compact.*

Corollary 2.2.12. *$[0, 1]^n$ is compact.*

Theorem 2.2.13. *$A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

Theorem 2.2.14. *$f : X \to \mathbb{R}$, f is continuous and X is compact. Then there exists $x_1, x_2 \in X$ such that $\inf_{x\in X} f(x) = f(x_1), \sup_{x\in X} f(x) = f(x_2)$.*

Proof. $f(X)$ is compact (why?), and thus $f(X)$ is closed and bounded. Because $f(X)$ is bounded, $l = \inf_{x \in X} f(x)$ and $u = \sup_{x \in X} f(x)$ exist ($\mathbb{R}$ is complete). Then l and u are limit points of $f(X)$ (by which theorem?). That $f(X)$ is closed means that $l, u \in f(X)$. □

2.3 Further Reading

5.1-5.4 in Mendelson.