SI112: Advanced Geometry		Spring 2018
	Lecture Note 6 — Mar. 15th, Thursday	
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1 Lecture 6

1.1 Overview of This Lecture

In previous lecture we introduced the notation of *topological equivalence* (1.2.1), and showed some examples (1.2.2, 1.2.3). Notice that *topological equivalence* relates two metric spaces, i.e., (X, d) and (Y, d'). It is natural to consider a special case, where $X = Y$. The corresponding theorems are 1.2.4, 1.2.6 and 1.2.8. However, does the concept, *topological equivalence*, even make sense? Theorem 1.2.12 gives a possible answer.

In the context of metric spaces, the various topological concepts such as continuity, neighborhood, and so on, may be characterized by means of open sets. Discarding the distance function and retaining the open sets of a metric space gives rise to a new mathematical object, called a *topological space* (1.2.14).

1.2 Proof of Things

Definition 1.2.1 (definition 7.6, chapter 2)**.** Two metric space (A, d_A) and (B, d_B) are said to be *topologically equivalent* or *homeomorphic* if there are inverse functions $f : A \to B$ and $g : B \to A$ such that f and g are continuous. In this event we say that the *topological equivalence is defined by f and g.*

Example 1.2.2 (homeomorphic spaces)**.** $X = \{0, 1\}, Y = \{0, 10\}$, $f : X \to Y$, $f(x) = 10x$, $g : Y \to X$, $g(y) = 0.1x$.

Example 1.2.3 (non-homeomorphic spaces)**.** (The explaination here for this example is from Real Mathematical Analysis) Consider the interval $[0, 2\pi) = \{x \in \mathbb{R} | 0 \leq x < 2\pi\}$ and define $f : [0, 2\pi) \to S^1$ to be the mapping $f(x) = (\cos x, \sin x)$, where S^1 is the unit circle in the plane, i.e., $S^1 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$. The mapping f is a continuous bijection, but the inverse bijection is not continuous. For there is a sequence of points (z_n) on S^1 in the fourth quadrant that converges to $p = (1, 0)$ from below, and $f^{-1}(z_n)$ does not converge to $f^{-1}(p) = 0$. Rather it converges to 2π. Thus, f is a continuous bijection whose inverse bijection fails to be continuous. See figure 1.

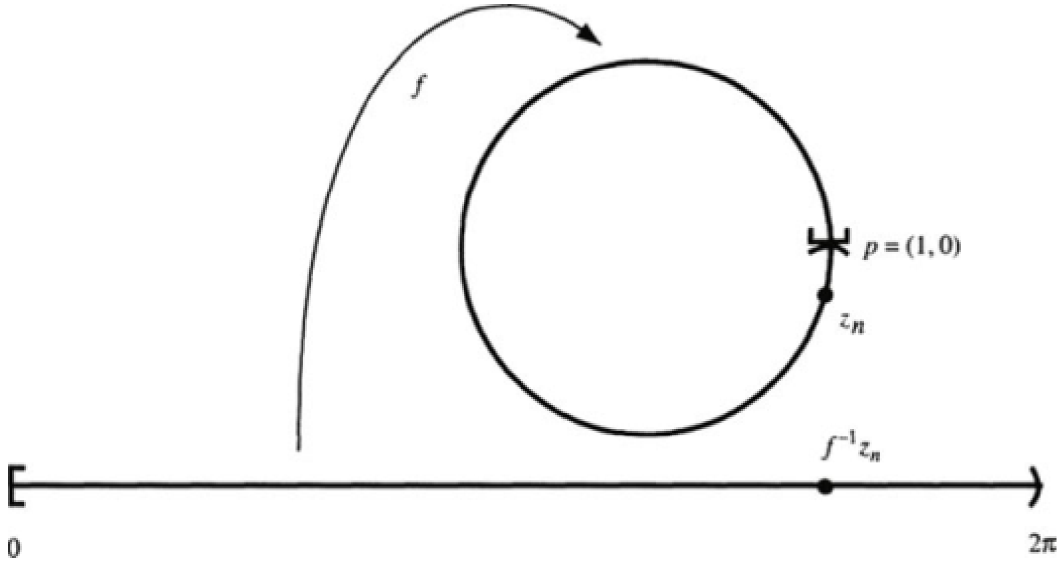

Figure 1: f wraps $[0, 2\pi)$ bijectively onto the circle.

Lemma 1.2.4 (lemma 7.8, chapter 2)**.** *Let (X, d) and (X, d') be two metric spaces. If there exists a number $K > 0$ such that for each $x, y \in X, d'(x, y) \leq Kd(x, y)$, then the identity mapping $i : (X, d) \to (X, d')$ is continuous.*

proof skeleton. Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{K}$. □

Exercise 1.2.5 (homework exercise)**.** Find an example of X, d_1, d_2 such that i is not continuous.

Corollary 1.2.6 (corollary 7.9, chapter 2)**.** *Let (X, d) and (X, d') be two metric spaces. If there exist positive numbers K and K' such that for each $x, y \in X$, we have*

$$d'(x, y) \leq Kd(x, y) \leq K'Kd'(x, y)\,,$$

then the identity mappings define a topological equivalence between (X, d) and (X, d').

proof skeleton. Simply apply lemma 1.2.4 twice. □

Example 1.2.7. Isomorphism of categories.

Corollary 1.2.8. $(\mathbb{R}^n, ||\cdot||_2) \sim (\mathbb{R}^n, ||\cdot||_1) \sim (\mathbb{R}^n, ||\cdot||_\infty)$, *where* $||x||_2 = \sqrt{x_1^2 + \cdots + x_n^2}, ||x||_1 = |x_1| + \cdots + |x_n|, ||x||_\infty = \max_{i=1,\ldots,n} x_i$.

Proof. It is easy to see from the definition of the norm that

$$||x||_\infty \le ||x||_2 \le \sqrt{n}||x||_\infty$$

and

$$||x||_\infty \le ||x||_1 \le n||x||_\infty\,,$$

from which

$$||x||_2 \le \sqrt{n}||x||_\infty \le \sqrt{n}||x||_1 \le \sqrt{n}n||x||_\infty \le \sqrt{n}n||x||_2$$

immediately follows. We finished the proof. □

Lemma 1.2.9 (lemma for theorem 1.2.12)**.** *If a function $f : X \to Y$ is injective, then for each subset O of X, $f^{-1}(f(O)) = O$.*

true proof. Let O be a subset of X. Then for each $x \in O$, we have $f^{-1}(f(x)) = \{x\}$ since f is injective. It follows that $f^{-1}(f(O)) = O$. □

Lemma 1.2.10 (lemma for theorem 1.2.12)**.** *Let (X, d_1) and (X, d_2) be two metric spaces. Let $f : X \to Y$ and $g : Y \to X$ be inverse functions, i.e., $gf = id_X, fg = id_Y$. Then for each subset O of X, we have $f(O) = g^{-1}(O)$.*

true proof. Given a subset O of X, we have $g(f(O)) = O$ and hence $g^{-1}(g(f(O))) = g^{-1}(O)$. That is, $f(O) = g^{-1}(O)$. □

Lemma 1.2.11 (lemma for theorem 1.2.12)**.** *Let $f : X \to Y$ be a function and O be a subset of Y, then we have $f^{-1}(O^C) = (f^{-1}(O))^C$.*

true proof. For each $x \in X$, we have

$$\begin{aligned} x \in f^{-1}(O^C) &\iff f(x) \in O^C \\ &\iff f(x) \notin O \\ &\iff x \notin f^{-1}(O) \\ &\iff x \in (f^{-1}(O))^C, \end{aligned} \tag{1}$$

which implies that $f^{-1}(O^C) = (f^{-1}(O))^C$. □

Theorem 1.2.12 (theorem 7.10, chapter 2)**.** *Let (X, d_1) and (X, d_2) be two metric spaces. Let $f : X \to Y$ and $g : Y \to X$ be inverse functions, i.e., $gf = id_X, fg = id_Y$. Then the following four statements are equivalent:*

1. *f and g are continuous;*

2. *A subset O of X is open if and only if $f(O)$ is an open subset of Y.*

3. *A subset F of X is closed if and only if $f(F)$ is a closed subset of Y.*

4. *For each $a \in X$ and subset N of X, N is a neighborhood of a if and only if $f(N)$ is a neighborhood of $f(a)$.*

true proof. We will prove this theorem in detail. The lemmas above will be extensively used.

(1 ⇒ 2) Assume that f and g are continuous. On the one hand, if $f(O)$ is an open subset of Y, then $O = f^{-1}(f(O))$ (lemma 1.2.9) is an open subset of X (f is continuous). On the other hand, if O is an open subset of X, then $f(O) = g^{-1}(O)$ (lemma 1.2.10) is open in Y (g is continuous), which completes the proof.

(2 ⇒ 1) It is left to you as an exercise.

(2 ⇒ 3) Suppose that (2) holds. On the one hand, if $f(F)$ is a closed subset of Y, which means $f(F)^C$ is open in Y, then, by lemma 1.2.10 and lemma 1.2.11,

$$f(F^C) = g^{-1}(F^C) = (g^{-1}(F))^C = (f(F))^C$$

is open in Y, which, by (2), means F^C is open in X and hence F is a closed subset of X. On the other hand, if F is closed in X, which means F^C is open in X and hence $f(F^C)$ is open in Y by (2). It follows that

$$(f(F))^C = (g^{-1}(F))^C = g^{-1}(F^C) = f(F^C)$$

is open in Y and hence $f(F)$ is a closed subset of Y, as desired.

(3 ⇒ 2) Immediate from above proof. It is left to you as an exercise.

(3 ⟺ 4) Unnecessarily verbose! We avoid this (Why can we skip it? Why is it unreasonable to prove this direction?).

(2 ⇒ 4) Suppose that (2) holds. Then for each $a \in X$ and $N \subset X$, N is a neighborhood of a if and only if N contains an open set O containing a if and only if $f(N)$ contains an open set $O' = f(O)$ containing $f(a)$ (since $a \in O \subset N \iff f(a) \in f(O) \subset f(N)$) if and only if $f(N)$ is a neighborhood of $f(a)$.

(4 ⇒ 1) Suppose that (4) holds. Then f is continuous, since for each $a \in X$ and each neighborhood $f(f^{-1}(U)) = U$ of $f(a)$, $f^{-1}(U)$ is a neighborhood of a. Similarly, g is continuous, since for each $b \in Y$ and each neighborhood V of $g(b)$, $g^{-1}(V) = f(V)$ is a neighborhood of $b = f(g(b))$.

□

Remark 1.2.13 (remark for theorem 1.2.12)**.** The proof for theorem 1.2.12 we give here is too long! It turns out that there is a simpler and therefore more elegant one.

simpler proof. This proof is based on the following observation. When proving

$$(1) \iff (2)\,,$$

we observe that if (1) holds, then, by lemma 1.2.9, one direction of (2) is what we've already proved (which direction?)! Can you transfer another direction, by again applying some lemmas above, into something we've already done before? The same is true for (1) if (2) holds. Exactly the same is again true for (1) $\iff$ (3) and for (1) $\iff$ (4).

(1 $\iff$ 2) Obvious.

(1 $\iff$ 3) Obvious.

(1 $\iff$ 4) Obvious.

we finished the proof. □

Definition 1.2.14 (definition 2.1, chapter 3)**.** Let X be a non-empty set and $\mathcal{J}$ a collection of subsets of X such that:

1. $X \in \mathcal{J}$.
2. $\emptyset \in \mathcal{J}$.
3. if $O_1, O_2, \ldots, O_n \in \mathcal{J}$, then $O_1 \cap O_2 \cap \cdots \cap O_n \in \mathcal{J}$.
4. If $O_i \in \mathcal{J}$ for each $i \in I$, then $\cup_{i \in I} O_i \in \mathcal{J}$.

The pair of objects $(X, \mathcal{J})$ is called a *topological space*. The set X is called the *underlying set*. The collection $\mathcal{J}$ is called the *topology* on the set X, and the members of $\mathcal{J}$ are called *open* sets.

Remark 1.2.15 (remark of definition 1.2.14)**.** This definition of topological space is in fact a theorem in metric space. Note that, here and in what follows, *open set* is nothing more than an element of the set $\mathcal{J}$. It is no longer (at least not now) a neighborhood of each of its points, and neighborhood is what we haven't defined yet. You have to forget the past to better start. We will eventually from this definition develop many theorems, which are what you've already been familiar with. Hence don't panic and stay tuned. Also note that our definition of topological space is in terms of open set. An alternative definition could be in terms of closed set. See the next example.

Example 1.2.16. Let $X = \mathbb{C}^n$. $Y \subset \mathbb{C}^n$ is defined to be closed if there exists $p_1, \ldots, p_l \in \mathcal{B}$, where $\mathcal{B} = \mathbb{C}[x_1, \ldots, x_n]$ is the ring of polynomial function on $\mathbb{C}^n$, such that

$$Y = \{z \in \mathbb{C}^n | p_1(z) = \cdots = p_l(z) = 0\} =: \mathbb{Z}(p_1, \ldots, p_l).$$

To show that the closed sets of $\mathbb{C}^n$ defined in this way give a topology on $\mathbb{C}^n$ (Zariski Topology), we need to show that

1. $\mathbb{C}^n$ is closed.
2. $\emptyset$ is closed.
3. The intersection of infinitely many Y_i is closed.
4. The union of finitely many Y_i is closed.

1.3 Further Reading

2.7, 3.1, 3.2 in Mendelson. Solve some problems in the book.