SI112: Advanced Geometry	Spring 2018
Lecture Note 5 — Mar. 13th, Tuesday	
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1 Lecture 5

1.1 Overview of This Lecture

Closed set and limit point introduced in previous lecture are somewhat difficult to understand. It is hard to believe closed set, as a complement of open set, has many great properties, while the definition of limit point seems unmotivated. It is essential to introduce some theorems (1.2.1,1.2.3), as a concretion of open/closed set and limit point, to see what they truly are on the real line. In the end of this lecture we take a half-hour quiz containing 3 problems.

1.2 Proof of Things

Lemma 1.2.1 (lemma 5.6, chapter 2). *Let b be the greatest lower bound of the non-empty subset A. Then, for each $\epsilon > 0$, there is an element $x \in A$ such that $x - b < \epsilon$.*

proof skeleton. Prove it by contradiction. □

Remark 1.2.2 (remark of lemma 1.2.1). Note that $S = [0, 1]$ ($S = (0, 1)$) is closed (open), 0 is the greatest lower bound of S, and it is also a limit point by definition. Also note that -0.0000001 is not a limit point of S. Is 0.7813 a limit point of S?

Corollary 1.2.3 (corollary 5.7, chapter 2). *Let b be a greatest lower bound of the non-empty subset A of real numbers. Then there is a sequence $a_1, a_2, \ldots$ of real numbers such that $a_n \in A$ for each n and $\lim_n a_n = b$.*

proof skeleton. Prove it by using lemma 1.2.1 and by constructing something like $(1, \frac{1}{2}, \ldots, \frac{1}{n})$. □

Remark 1.2.4 (remark of corollary 1.2.3). Try to connect this corollary to the definition of limit point.

Definition 1.2.5 (definition 5.8, chapter 2). Let (X, d) be a metric space. Let $a \in X$ and Let A be non-empty subset of X. The greatest lower bound of the set of numbers of the form $d(a, x)$ for $x \in A$ is called the *distance between a and A* and is denoted by $d(a, A)$.

Question 1.2.6. What's $d(a, A)$ for $A = [0, 1], a = 0.3$, for $A = [0, 1], a = -0.3$ and for $A = (0, 1), a = 0$?

Corollary 1.2.7 (corollary 5.9, chapter 2)**.** *Let (X, d) be a metric space, $a \in X$, and A a non-empty subset of X. Then there is a sequence $a_1, a_2, \dots$ of points of A such that* $\lim_n d(a, a_n) = d(a, A)$.

proof skeleton. Use corollary 1.2.3 and definition 1.2.5. □

Exercise 1.2.8. Let (X, d) be a metric space, $A \subset X$. Prove or give a counterexample: for $a \in X$, $d(a, A) = 0$ if and only if $a \in A$ or a is a limit point of A.

Theorem 1.2.9 (theorem 6.9, chapter 2)**.** *A subset F of a metric space (X, d) is closed if and only if for each point $x \in X$, $d(x, F) = 0$ implies $x \in F$.*

proof skeleton. Recall that we've proved in the previous lecture that a set F is said to be closed if and only if F contains all its limit points. It is enough to show that F contains all its limit points if and only if for each point $x \in X$, $d(x, F) = 0$ implies $x \in F$, which follows from the exercise 1.2.8. □

Theorem 1.2.10 (theorem 6.10, chapter 2)**.** *Let $(X, d), (Y, d')$ be metric spaces. A function $f : X \to Y$ is continuous if and only if for each closed subset A of Y, the set $f^{-1}(A)$ is closed subset of X.*

proof skeleton. Immediate from open set characterization of continuity and $C(f^{-1}(A)) = f^{-1}(C(A))$. □

1.3 Further Reading

2.5, 2.6, 2.7 in Mendelson. Solve some problems in the book.