SI112: Advanced Geometry	Spring 2018
Lecture Note 4 — Mar. 8th, Thursday	
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1 Lecture 4

"*We will have a quiz on Monday.*"

1.1 Overview of This Lecture

we discussed some examples of open ball, emphasizing that an open ball might not look like a ball at all (1.2.1). Keeping in the mind the notation of open set, an abstraction of open ball, it is natural to consider the complement of the open set, which we define as *closed set* (1.2.3). The definition of open set and closed set will lead to many important consequences, some of which (1.2.4, 1.2.9, 1.2.10) are explored in this lecture.

1.2 Proof of Things

Example 1.2.1. The "shape" of an open ball,

$$B(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\},$$

where (X, d) is a metric space, depends on the metric d and the underlying space X. the name of open ball stems from $(\mathbb{R}^2, d)$ $(d(x, y) = ||x - y||_2)$, where an open ball looks like exactly a ball. However, it is not always the case.

If X itself doesn't contain a ball (e.g., $\mathbb{R}$), an open ball in X is of course doesn't look like a ball. Even if X contains balls (e.g., $X = \mathbb{R}^n$), an intentionally-designed metric d can make an open ball "not a ball" (e.g., discrete metric on any set X, $d(x, y) = ||x - y||_1$ on $\mathbb{R}^n$).

Exercise 1.2.2. Show that $d(A, B) = rank(A - B)$ is a metric on $\mathbb{R}^{m \times n}$.

Definition 1.2.3 (closed set, definition 6.5, chapter 2)**.** A subset F of a metric space is said to be *closed* if its complement, F^C, is open.

Definition 1.2.4 (limit point, definition 6.6, chapter 2)**.** Let A be a subset of a metric space X. A point $b \in X$ is called a *limit point* of A if every neighborhood of b contains a point of A different from b.

Exercise 1.2.5. Thinking of the intervals on the real line. Relates them to open set, closed set, and limit point. For example, given an interval $I = (0, 1)$, which is open. Then ask yourself, "is I open or closed? What's the limit points of I? Is 1.000000001 a limit point of I? How about 0.99999992317864?" Ask yourself the similar questions for the intervals $M = (0, 1], N = [0, 1]$.

Remark 1.2.6 (remark of limit point)**.** The definition of *limit point* is somewhat weird. Here is my understanding.

1. First notice the term "every neighborhood". In the last lecture we've seen a description like this (see lecture note 3). We said that a convergent sequence $a_1, a_2, \ldots$ with limit a will eventually falls into any specified neighborhood of a. In other words, for "every neighborhood" V_a of a, the sequence $a_1, a_2, \ldots$ will eventually falls into V_a. That might be why we call it *limit point*.

2. Then we look at the entire definition. What does it mean that every neighborhood of b contains a point of A different from b. We are too lazy to care about *every neighborhood* of b. Can we have a neighborhood of b, which is, somewhat, *smallest*, contains a point of A different from b, and is included by all of other neighborhoods of b (we've known that if Q is a neighborhood of a and $Q \subset P$, then P is also a neighborhood of a)? In this way we may define *limit point* as "b is a limit point of A if the smallest neighborhood of b contains a point of A different from b". The answer is "No, we can not". Why?

Lemma 1.2.7. $A \cap B = \emptyset \iff A \subset B^C$.

proof skeleton. Immediate but seemingly irrelevant. Try to prove it both formally and graphically. □

Remark 1.2.8 (remark of the lemma 1.2.7)**.** To prove the very first theorem 1.2.9 which relates closed set and limit point, we have no choice but use their definition. Closed set (B) is the complement of an open set (B^C). Open set (B^C) contains a neighborhood (A) of each of its points. Do you see it?

Theorem 1.2.9 (theorem 6.7, chapter 2)**.** *A subset F of X is closed if and only if F contains all limit points.*

proof skeleton. Try to prove it yourself by using the lemma 1.2.7. Do not read the proof in the book, which might make you cry. Read my proof below after some trials. In this proof

$N \in \mathcal{N}_a$ denotes that N is a neighborhood of a.

$$\begin{aligned} F \text{ is closed.} &\iff F^C \text{ is open.} \\ &\iff \forall a \in F^C, \exists N \in \mathcal{N}_a \text{ such that } a \in N \subset F^C. \\ &\iff \forall a \in F^C, \exists N \in \mathcal{N}_a \text{ such that } a \in N \text{ and } N \cap F = \emptyset. \\ &\iff \forall a \in F^C,\ a \text{ is not a limit point of } F. \\ &\iff \text{all limit points of } F \text{ are contained in } F. \end{aligned} \tag{1}$$

□

Theorem 1.2.10 (theorem 6.8, chapter 2)**.** *In a metric space (X, d), a set $F \subset X$ is closed if and only if for each sequence $a_1, a_2, \ldots$ of points of F that converges to a point $a \in X$ we have $a \in F$.*

proof skeleton. When proving this theorem, you may invoke theorem 1.2.9, which is the first theorem we've proved about closed set and we do not want to go back to the its definition to prove other new theorems.

On the one hand, let F be closed. You may suppose there is a sequence $a_1, a_2, \cdots \in F$ and it converges to a (if there are no such sequences, you are done) and you have to prove $a \in F$. There are then two cases, the set $\{a_1, a_2, \ldots\}$ is finite, or it is an infinite set. Deal with them seperately.

On the other hand, you need to show F is closed if ...... what? □

1.3 Further Reading

2.5, 2.6 in Mendelson. Solve some problems in the book.